DISCRETE MATHEMATICS

ELSEVIER Discrete Mathematics 175 (1997) 211–219

Equations in the theory of Q-distributive lattices

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Received 12 May 1995; revised 13 February 1996

Abstract

A *Q-distributive lattice* is an algebra $(L, \vee, \wedge, \nabla, 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and ∇ satisfies the equations $\nabla 0 = 0$, $x \wedge \nabla x = x$, $\nabla(x \vee y) = \nabla x \vee \nabla y$ and $\nabla(x \wedge \nabla y) = \nabla x \wedge \nabla y$. The aim of this paper is to find, for each proper subvariety of the variety of Q-distributive lattices, an equation which determines it, relatively to the whole variety, as well as to give a characterization of the minimum number of variables needed in such equation.

A quantifier on a bounded distributive lattice L is a unary operation ∇ on L that satisfies $\nabla 0 = 0$, $x \wedge \nabla x = x$, $\nabla(x \vee y) = \nabla x \vee \nabla y$ and $\nabla(x \wedge \nabla y) = \nabla x \wedge \nabla y$. A quantifier ∇ is called *simple* if and only if it is given by the prescription: $\nabla 0 = 0$ and $\nabla a = 1$ for each $a \neq 0$.

A *Q-distributive lattice* is an algebra $(L, \vee, \wedge, \nabla, 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and ∇ is a quantifier on L. The variety of Q-distributive lattices will be denoted by $\mathcal{Q}$.

Q-distributive lattices were introduced by Cignoli in [2] and he showed that the lattice of equational subclasses of $\mathcal{Q}$ is a chain of type $\omega + 1$.

In this paper we first find, for each proper subvariety of $\mathcal{Q}$, an equation which determines it. Secondly, we give a characterization of the minimum number of variables needed in an equation characterizing a given subvariety, and we determine this number in some cases.

We begin with some notation. We shall denote by $\mathbf{2}$ the Boolean algebra with two elements. For each natural number p, B_p will denote the Boolean algebra $\mathbf{2}^p$ endowed with the simple quantifier, and C_p the lattice $\mathbf{2}^p$ with a new 1 added endowed with the simple quantifier, provided $p \geqslant 1$, and $C_0 = B_0$. For each $(p, q) \in w \times w$, D_{pq} will denote the subjacent lattice $B_p \times C_q$ endowed with the simple quantifier. (Observe, in

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PII S0012-365X(96)00151-3

particular, that D_{00} is a 1-element algebra, D_{p0} is Q-isomorphic to B_p, and that D_{0p} is Q-isomorphic to C_p.) The subvariety of $\mathcal{Q}$ generated by the algebra D_{pq} will be denoted by $\mathcal{D}_{pq}$.

Cignoli showed in [2] that D_{pq} is a subdirecty irreducible algebra in $\mathcal{Q}$, for each $p, q \in \omega$ and that the lattice of equational subclasses is a chain of type $\omega + 1$ under inclusion:

$$\begin{aligned}
&\mathcal{D}_{00} \subset \\
&\mathcal{D}_{10} \subset \mathcal{D}_{01} \subset \\
&\mathcal{D}_{20} \subset \mathcal{D}_{02} \subset \mathcal{D}_{11} \subset \\
&\mathcal{D}_{30} \subset \mathcal{D}_{03} \subset \mathcal{D}_{12} \subset \mathcal{D}_{21} \subset \\
&\mathcal{D}_{40} \subset \mathcal{D}_{04} \subset \mathcal{D}_{13} \subset \mathcal{D}_{22} \subset \mathcal{D}_{31} \subset \\
&\ldots
\end{aligned}$$

Theorem 1. (i) *For each natural number $p \neq 0$, the variety $\mathcal{D}_{p0}$ is characterized by the equation*:

$$(I_p)\ \left(\bigwedge_{k=1}^{p} \nabla x_k\right) \wedge \left[\left(\bigvee_{1 \leqslant i<j \leqslant p} \nabla(x_i \wedge x_j)\right) \vee \left(\bigvee_{i=1}^{p} x_i\right)\right] = \bigwedge_{k=1}^{p} \nabla x_k .$$

(ii) *For each natural number p, the variety $\mathcal{D}_{p1}$ is characterized by the equation*:

$$(I_{p1})\ \left(\bigwedge_{k=1}^{p+2} \nabla x_k\right) \wedge \left[\bigvee_{1 \leqslant i<j \leqslant p+2} \nabla(x_i \wedge x_j)\right] = \bigwedge_{k=1}^{p+2} \nabla x_k .$$

(iii) *For each natural number p and each natural number $q \geqslant 2$, the variety $\mathcal{D}_{pq}$ is characterized by the equation*:

$$\begin{aligned}
(I_{pq})\ &\left(\bigwedge_{k=1}^{p+q} \nabla x_k\right) \wedge [t(x_1, x_2, \ldots, x_{p+q+1}) \vee x_{p+q+1}] \\
&= \left(\bigwedge_{k=1}^{p+q} \nabla x_k\right) \wedge \left[t(x_1, x_2, \ldots, x_{p+q+1}) \vee \left(x_{p+q+1} \wedge \left(\bigvee_{k=p+2}^{p+q} x_k\right)\right)\right],
\end{aligned}$$

where

$$t(x_1, x_2, \ldots, x_{p+q+1}) = \left(\bigvee_{1 \leqslant i<j \leqslant p+q} \nabla(x_i \wedge x_j)\right) \vee \left(\bigvee_{i=1}^{p+1} \nabla(x_{p+q+1} \wedge x_i)\right).$$

Proof. If I stands for an identity and $A \in \mathcal{Q}$, we will express the fact that I holds in A symbolically by $A \models I$, otherwise we write $A \not\models I$. It is plain that the subvarieties of $\mathcal{Q}$ defined by the identities I_p, I_{p1} and I_{pq} are proper ones. Hence, they are of the form $\mathcal{D}_{rs}$, for some $r, s \in w$. Moreover, since the varieties $\mathcal{D}_{pq}$ form a chain and they are generated by the algebras D_{pq}, it is enough to show that $D_{p0} \models I_p$, $D_{0p} \not\models I_p$, $D_{p1} \models I_{p1}$, $D_{p+20} \not\models I_{p1}$, $D_{pq} \models I_{pq}$ and $D_{p+1q-1} \not\models I_{pq}$, provided $q \geqslant 2$.

Recall that x and y are said to be *disjoint* if $x \wedge y = 0$.

(i) Let A be an algebra in $\mathscr{Q}$ endowed with the simple quantifier. Then it is easy to check that $A \models I_p$ if and only if A satisfies the following property:

(1) If $\{a_1, a_2, \ldots, a_p\}$ are p elements of A distincts from 0 and pairwise disjoint, then $a_1 \vee a_2 \ldots \vee a_p = 1$.

It is easy to see that a finite bounded distributive lattice L satisfies (1) if and only if either it has at most $p-1$ atoms and in this case (1) holds trivially or it is a Boolean algebra with p atoms. Therefore, $D_{p0} \models I_p$ and $D_{0p} \not\models I_p$ and the proof of (i) is complete.

(ii) Let A be an algebra in $\mathscr{Q}$ endowed with the simple quantifier. It is easy to check that $A \models I_{p1}$ if and only if A satisfies the following property:

There exists no set $\{a_1, a_2, \ldots, a_{p+2}\}$ of $p+2$ nonzero and pairwise disjoint elements of A.

Therefore, since D_{p1} has $p+1$ atoms and D_{p+20} has $p+2$ atoms, we infer that $D_{p1} \models I_{p1}$ and $D_{p+20} \not\models I_{p1}$ and the proof is complete.

We turn now our attention to the general case:

(iii) Let A be an algebra in $\mathscr{Q}$ endowed with the simple quantifier. It is again easy to see that $A \models I_{pq}$ if and only if A satisfies the following property:

(n) If $\{a_1, a_2, \ldots, a_{p+q}\}$ is a set of $p+q$ nonzero and pairwise disjoint elements of A, and $b \wedge a_i = 0$ for all $i \in \{1, 2, \ldots, p+1\}$, then $b \leqslant a_{p+2} \vee a_{p+3} \ldots \vee a_{p+q}$.

We are going to denote by $\{c_1, c_2, \ldots, c_{p+q}\}$ the set of atoms of D_{pq} and by d the only join irreducible that is not an atom of D_{pq}. Note that $d = (0, 1)$, where 1 is the last element of C_q. We claim that D_{pq} satifies (n). Indeed, let $\{a_1, a_2, \ldots, a_{p+q}\}$ be a set of $p+q$ elements of D_{pq} different from 0 and pairwise disjoint, and let $b \in D_{pq}$ be such that b is disjoint with a_i for all $i \in \{1, 2, \ldots, p+1\}$. Since D_{pq} has exactly $p+q$ atoms, we infer that

(a) for each $i \in \{1, 2, \ldots, p+q\}$ there is a unique $j \in \{1, 2, \ldots, p+q\}$ such that $c_j \leqslant a_i$.

We may suppose, without loss of generality, that $c_i \leqslant a_i$ for all $i \in \{1, 2, \ldots, p+q\}$. Suppose that $c_i < a_i$ for some $i \in \{1, 2, \ldots, p+q\}$. Suppose first that $c_i = (0, \alpha_i)$, where α_i is an atom of C_q, while $a_i = (x, y)$, where $\alpha_i \leqslant y$. Since $q \geqslant 2$, then $(0, 1)$ is greater than at least two atoms. Hence, we infer from condition (a) that $y \neq 1$. Therefore $y \in \mathbf{2}^q$. Now $\alpha_i \neq y$ would imply $\alpha_j < y$ for some atom $\alpha_j \neq \alpha_i$, hence $c_j = (0, \alpha_j) \leqslant a_i$, a contradiction; similarly $x \neq 0$ would imply $\beta_k \leqslant x$ for some atom β_k of B_p, hence $c_k = (\beta_k, 0) \leqslant a_i$, again a contradiction. The case $c_i = (\alpha_i, 0)$, where α_i is an atom of B_p is even simpler since B_p is a Boolean algebra. Hence, $c_i = a_i$ for $i \in \{1, 2, \ldots, p+q\}$. Suppose that $d \leqslant b$. Since $b \wedge a_i = b \wedge c_i = 0$ for all $i \in \{1, \ldots, p+1\}$, then d would be disjoint with $p+1$ atoms, which is in contradiction with the fact that d is greater than exactly q atoms. Therefore, b is the join of a set of atoms contained in $\{c_{p+2}, \ldots, c_{p+q}\} = \{a_{p+2}, \ldots, a_{p+q}\}$ and hence, $b \leqslant a_{p+2} \cdots \vee a_{p+q}$.

Therefore $D_{pq} \models I_{pq}$. To prove that $D_{p+1q-1} \not\models I_{pq}$, note that the only irreducible which is not an atom of D_{p+1q-1} is disjoint with $p+1$ atoms and greater than the others $q-1$ atoms. Therefore, D_{p+1q-1} does not satisfy (n) and the proof is complete. □

Note that for each $p, q \in \omega$, $q \neq 0$, the equations I_p and I_{pq} have p and $p+q+1$ variables, respectively. We are going to show that in most cases, the varieties $\mathscr{D}_{p0}$ and $\mathscr{D}_{pq}$ can be determined by an equation having less than p and $p+q+1$ variables, respectively.

Recall that an algebra A in a variety $\mathscr{V}$ is said to be a *splitting algebra* in $\mathscr{V}$ if there exists a subvariety $\mathscr{W}$ of $\mathscr{V}$ such that $A \notin \mathscr{W}$ and for any subvariety $\mathscr{W}'$ of $\mathscr{V}$ either $A \in \mathscr{W}'$ or $\mathscr{W}'$ is a subvariety of $\mathscr{W}$, i.e. there is a largest subvariety $\mathscr{W}$ of $\mathscr{V}$ not containing A. The notion of splitting algebra was introduced by McKenzie in [7] when $\mathscr{V}$ is the variety of lattices. He showed that if A is finite, subdirectly irreducible and generated by a set of n elements, then $\mathscr{W}$ can be characterized by an equation, called the *conjugate equation*, having at most n variables (see [7, p. 19]; [4, Lemma 3.6], see also [3]). Moreover, the following result is of independent interest.

Theorem 2. *Let $\mathscr{V}$ be a congruence distributive variety such that every proper subvariety is generated by a single finite subdirectly irreducible algebra and the lattice of equational subclasses of $\mathscr{V}$ form a chain of type $\omega + 1$. Then, for each proper subvariety $\mathscr{W}$ of $\mathscr{V}$ the following conditions hold*:

(a) *$\mathscr{W}$ can be determined, relative to $\mathscr{V}$, by the addition of a single equation.*

(b) *The minimum number of variables needed in an equation characterizing the equational class $\mathscr{W}$ coincides with the minimum number of generators needed to generate A_W, where A_W is the finite subdirectly irreducible algebra that generates the succesor of $\mathscr{W}$.*

Proof. If K is any class of algebras then $V(K)$ will denote the variety generated by K. Let $\mathscr{W}$ be a proper subvariety of $\mathscr{V}$. Since the lattice $\mathbf{L}(\mathscr{V})$ of equational subclasses of $\mathscr{V}$ form a chain of type $\omega + 1$, it follows that the algebra A_W that generates the succesor of $\mathscr{W}$ in $\mathbf{L}(\mathscr{V})$ is a splitting algebra. Indeed, $\mathscr{W}$ is the largest subvariety of $\mathscr{V}$ not containing A_W. Therefore, $\mathscr{W}$ satisfies condition (a). To prove that $\mathscr{W}$ satisfies (b), let n be the minimum number of generators needed to generate A_W. Let α be an equation that characterizes $\mathscr{W}$ having at most n variables, say $x_1, \dots, x_r$, $r \leqslant n$. Suppose that $r < n$. Since $A_W \notin \mathscr{W}$, then there are r elements $a_1, \dots, a_r$ in A_W which do not satisfy α. It follows that the subalgebra B generated by $a_1, \dots, a_r$ is a proper subalgebra of A. Since B is a finite algebra, then it is isomorphic to a subdirect product of a finite number of subdirectly irreducible finite algebras in $V(B)$, say $B_1, \dots, B_m$. Hence, $V(B) = V(B_1, \dots, B_m)$. Suppose that $V(B) = V(A_W)$. Since A_W is subdirectly irreducible and $\mathscr{V}$ is a congruence distributive variety, we can apply the well-known results of Jónsson [5] to obtain that

(∗) A_W is a homomorphic image of a subalgebra of B_i for some $i \in \{1, \dots, m\}$.

On the other hand, since B is a proper subalgebra of A_W and B_i is a homomorphic image of B, we infer that the number of elements of B_i is less than the number of elements of A_W, which is in contradiction with $(*)$. Therefore, $V(B)$ is a proper subvariety of $V(A_W)$, and then $B \in \mathscr{W}$ and we again obtain a contradiction, since α does not hold in B. Therefore, $r = n$ and the proof is complete. $\square$

Therefore, $\mathscr{Q}$ satisfies the hypotheses of Theorem 2. Note that since D_{pq} is endowed with the simple quantifier, for each $S \subseteq D_{pq}$, the Q-distributive lattice generated by S coincides with the bounded distributive lattice generated by S.

Let L be a finite distributive lattice and $G = \{b_1, b_2, \ldots, b_n\}$ be a set of generators of L. For each $p \in L$, let $A_p = \{i \in \{1,2,\ldots,n\} \mid p \leqslant b_i\}$. If p is a join irreducible element of L, then it is easy to check that $p = \bigwedge_{i \in A_p} b_i$. If j is any index and a_j is a joint irreducible element of L, we shall write A_j instead of A_{a_j}.

Lemma 3. *Let n, k be natural numbers with $k \leqslant n$, and let $p = \binom{n}{k}$. Then there exists a set of generators G of D_{p0} with n elements satisfying the following condition:*

(c) *For each atom a of D_{p0} there are exactly k elements of G that include it and the meet of those elements equals a.*

Proof. Let $\{a_1, a_2, \ldots, a_p\}$ and $\{p_1, p_2, \ldots, p_n\}$ be the set of atoms of D_{p0} and $\mathbf{2}^n$, respectively. If $T_p = \{x \in \mathbf{2}^n$ such that x is a join of k-atoms$\}$, then T_p has p elements, say $l_1, l_2, \ldots, l_p$. Next define, for each $i \in \{1,2,\ldots,n\}$, $P_i = \{j \in \{1,2,\ldots,p\} \mid l_j \geqslant p_i\}$ and $b_i = \bigvee_{j \in P_i} a_j$. We assert that $b_1, b_2, \ldots, b_n$ generate the distributive lattice D_{p0}. Indeed, let $i \in \{1,2,\ldots,p\}$. Then there are k atoms $p_{i_1}, p_{i_2}, \ldots, p_{i_k}$ of $\mathbf{2}^n$ such that $l_i = p_{i_1} \vee p_{i_2} \cdots \vee p_{i_k}$. Hence, $i \in P_{i_1} \cap \cdots \cap P_{i_k}$, and then $a_i \leqslant b_{i_1} \wedge \cdots \wedge b_{i_k}$. Let $j \in \{1,2,\ldots,p\}$ be such that $a_j \leqslant b_{i_1} \wedge \cdots \wedge b_{i_k}$. Then $l_j \geqslant p_{i_1} \vee \cdots \vee p_{i_k}$. Since l_j is a join of k atoms we infer that $l_j = p_{i_1} \vee \cdots \vee p_{i_k} = l_i$. Then $i = j$. Thus, we have shown that $b_{i_1} \wedge \cdots \wedge b_{i_k} = a_i$. Hence, every atom of D_{p0} is a meet of k elements of the set $G = \{b_1, b_2, \ldots, b_n\}$. Therefore, $b_1, b_2, \ldots, b_n$ generate D_{p0}. To show that G satisfies condition (c), let $j \in \{1,\ldots,p\}$. Since $a_j = \bigwedge_{i \in A_j} b_i$, it is enough to show that A_j has k elements. By taking into account the definition of the elements b_i, it is easy to check that $A_j = \{i \in \{1,2,\ldots,n\} \mid p_i \leqslant l_j\}$. Since l_j is a join of k atoms, we infer that A_j has k elements and the proof is complete. $\square$

Lemma 4. *Let $p, q \in \omega$ and let n be the smallest natural number such that $p+q \leqslant \binom{n}{[\frac{1}{2}n]}$, where $[x]$ denotes the integral part of the number x. Then D_{pq} cannot be generated with less than n elements.*

Proof. Suppose that D_{pq} is generated with r elements $c_1, c_2, \ldots, c_r$ with $r < n$. It follows that for each $i \in \{1,2,\ldots,p+q\}$, $a_i = \bigwedge_{j \in A_i} c_j$, where $a_1, a_2, \ldots, a_{p+q}$ are the atoms of D_{pq}. Since a_i is an atom for each $i \in \{1, 2, \ldots, p+q\}$, we have that the sets $A_1, A_2, \ldots, A_{p+q}$ form an antichain of subsets of $\{1,2,\ldots,r\}$ and by a well-known theorem of Sperner, this implies that $p+q \leqslant \binom{r}{[\frac{1}{2}r]}$ in contradiction with the hypothesis. □

Theorem 5. *Let $p \in \omega$ and let n be the smallest natural number such that $p \leqslant \binom{n}{[\frac{1}{2}n]}$. Then the minimum number of generators needed to generate the bounded distributive lattices D_{p0} and D_{0p} is n.*

Proof. Let $q = \binom{n}{[\frac{1}{2}n]}$. It follows from Lemma 3 that D_{q0} is generated with n elements $b_1, b_2, \ldots, b_n$. Let $h : D_{q0} \to D_{p0}$ be a Boolean algebra epimorphism. Since h is surjective we infer that $h(b_1), h(b_2), \ldots, h(b_n)$ generate D_{p0}. Suppose that there are $i, j \in \{1, 2, \ldots, n\}$ such that $h(b_i) = h(b_j)$ and $i \neq j$. Then, D_{p0} would be generated with less than n elements which is in contradiction with Lemma 4. Therefore, D_{p0} is generated with n elements and we infer again from Lemma 4 that D_{p0} cannot be generated with less than n elements. The same argument shows that the minimum number of generators needed to generate D_{0p} is also n, since 1 is a nullary operation. □

Theorem 6. *Let $p, q \in \omega$, $p, q \neq 0$ and let n be the smallest natural number such that $p+q \leqslant \binom{n}{[\frac{1}{2}n]}$. Then*

(a) *The minimum number of generators needed to generate D_{pq} is either n or $n+1$.*

(b) *If $p+q = \binom{n}{[\frac{1}{2}n]}$, and $q > 1$, then the minimum number of generators needed to generate D_{pq} is n if and only if either there is $t \in \{0, 1, \ldots, [\frac{1}{2}n]-1\}$ such that $q = \binom{n-t}{[\frac{1}{2}n]-t}$ or there is $t \in \{0, 1, \ldots, n-[\frac{1}{2}n]-1\}$ such that $q = \binom{n-t}{n-[\frac{1}{2}n]-t}$.*

(c) *If $p+1 = \binom{n}{[\frac{1}{2}n]}$, then the minimum number of generators needed to generate D_{p1} is $n+1$.*

Proof. (a) By taking into account Theorem 5, we infer that $B_p \times B_q$ can be generated with n elements $b_1, b_2, \ldots, b_n$. Let d be the only join irreducible that is not an atom of $D_{pq} = B_p \times C_q$. Then it is obvious that $b_1, b_2, \ldots, b_n, d$ generate D_{pq} and by Lemma 4 we infer that D_{pq} cannot be generated with less than n elements.

(b) Suppose first that D_{pq} is generated with n elements $b_1, b_2, \ldots, b_n$. Let $\{a_1, a_2, \ldots, a_{p+q}\}$ be the set of atoms of D_{pq} and let d be the only join irreducible that is not atom of D_{pq}. Since d is greater than q atoms, we may assume, without loss of generality, that $d > a_{p+1} \vee a_{p+2} \vee \cdots \vee a_{p+q}$. It follows that the sets $A_1, A_2, \ldots, A_{p+q}$ form an antichain of subsets of $\{1, 2, \ldots, n\}$ such that $A_d \subseteq \bigcap_{p+1 \leqslant i \leqslant p+q} A_i$. Moreover, since $a_i \nleqslant d$ for each $i \in \{1, \ldots, p\}$, we have that $A_d \nsubseteq A_i$ for each $i \in \{1, 2, \ldots, p\}$. On the other hand, since $p+q = \binom{n}{[\frac{1}{2}n]}$, it follows again from Sperner's theorem that the sets $A_1, A_2, \ldots, A_{p+q}$ form an antichain of maximal size, and that this antichain coincides either with the set of all subsets of $\{1, 2, \ldots, n\}$ of cardinal $[\frac{1}{2}n]$ or with the set of all subsets of $\{1, 2, \ldots, n\}$ of cardinal $n - [\frac{1}{2}n]$. Suppose first that A_i has $[\frac{1}{2}n]$ elements for all $i \in \{1, 2, \ldots, p+q\}$. Let t be the cardinal of $A_{p+1} \cap A_{p+2} \cdots \cap A_{p+q}$. It is plain that $t \leqslant [\frac{1}{2}n] - 1$. Let $i \in \{1, 2, \ldots, p\}$. Since $A_d \nsubseteq A_i$ and $A_d \subseteq \bigcap_{p+1 \leqslant i \leqslant p+q} A_i$, it follows that $A_{p+1} \cap A_{p+2} \cdots \cap A_{p+q} \nsubseteq A_i$. Therefore, we infer that $\{r_1, r_2, \ldots, r_t\} = A_{p+1} \cap A_{p+2} \cdots \cap A_{p+q}$ is contained in exactly q sets of cardinality $[\frac{1}{2}n]$ and by a combinatorial argument it is easy to see that

$$q = \binom{n-t}{[\frac{1}{2}n] - t}.$$

Analogously, we infer that if A_i has $n - [\frac{1}{2}n]$ elements for $i = 1, 2, \ldots, p+q$, then there is $t \in \{0, 1, \ldots, n - [\frac{1}{2}n] - 1\}$ such that

$$q = \binom{n-t}{n - [\frac{1}{2}n] - t}.$$

For the converse, suppose first that $q = \binom{n-t}{[\frac{1}{2}n]-t}$ for some $0 \leqslant t \leqslant [\frac{1}{2}n] - 1$. Let $G = \{b_1, b_2, \ldots, b_n\}$ be a set of generators of $B_p \times B_q$ defined as in Lemma 3, where in this case $k = [\frac{1}{2}n]$. It is easy to check that if $\{b_{i_1}, b_{i_2}, \ldots, b_{i_t}\}$ is any subset of G of t elements, then $b_{i_1} \wedge \cdots \wedge b_{i_t}$ is a join of q atoms. We may assume without loss of generality, that:

(1) $b_1 \wedge \cdots \wedge b_t = a_{p+1} \vee \cdots \vee a_{p+q}$.

Now we are going to define elements $b'_1, b'_2, \ldots, b'_n$ of D_{pq} as follows: $b'_i = b_i$ if $i \neq 1, 2, \ldots, t$ and $b'_1 = b_1 \vee d, b'_2 = b_2 \vee d, \ldots, b'_t = b_t \vee d$. We claim that the elements $b'_1, \ldots, b'_n$ generate the algebra D_{pq}. Indeed, it is plain that $b'_1 \wedge \cdots \wedge b'_t = d$. On the other hand, it follows from (1) that for each $i \in \{1, \ldots, q\}$, $a_{p+i} \leqslant b_1 \wedge \cdots \wedge b_t$. By Lemma 3, we have that a_{p+i} is the meet of the $[\frac{1}{2}n]$ elements of G that include a. Hence, we infer that there are $[\frac{1}{2}n] - t$ elements $b_{i_{t+1}}, \ldots, b_{i_{[\frac{n}{2}]}} \in \{b_{t+1}, \ldots, b_n\}$ such that $a_{p+i} = b_1 \wedge \cdots \wedge b_t \wedge b_{i_{t+1}} \cdots \wedge b_{i_{[\frac{n}{2}]}}$. From this equality and (1) we infer that $d \wedge b_{i_{t+1}} \cdots \wedge b_{i_{[\frac{n}{2}]}} = a_{p+i}$ and it is easy to check that this implies $b'_1 \wedge \cdots \wedge b'_t \wedge b'_{i_{t+1}} \wedge \cdots b'_{i_{[\frac{n}{2}]}} = a_{p+i}$. To complete the proof, it remains to show that a_i is a meet of elements of the set $\{b'_1, b'_2, \ldots, b'_n\}$ for each $i \in \{1, 2, \ldots, p\}$. From Lemma 3 we have that there are $i_1, i_2, \ldots, i_{[n/2]} \in \{1, 2, \ldots, n\}$ such that $a_i = b_{i_1} \wedge \cdots \wedge b_{i_{[n/2]}}$. Since $i \notin \{p+1, \ldots, p+q\}$ it follows that $a_i \nleqslant b_1 \wedge \cdots \wedge b_t$. Hence, the set $T = \{j \in \{1, 2, \ldots, [\frac{n}{2}]\} \mid i_j \in \{1, 2, \ldots, t\}\}$ has less than t elements, say $j_1, \ldots, j_r$. Therefore $b'_{i_1} \wedge \cdots \wedge b'_{i_{[\frac{n}{2}]}} = (b_{i_{j_1}} \vee d) \wedge \cdots \wedge (b_{i_{j_r}} \vee d) \wedge \bigwedge_{j \notin T} b_{i_j} = a_i \vee (d \wedge \bigwedge_{j \notin T} b_{i_j})$. We assert that $d \wedge \bigwedge_{j \notin T} b_{i_j} = 0$. Indeed, first note that if c is an element of D_{pq} then $d \wedge c = 0$ iff $a_{p+i} \wedge c = 0$ for each $i \in \{1, 2, \ldots, q\}$, which is equivalent to $b_1 \wedge \cdots \wedge b_t \wedge c = 0$. On the other hand, we claim that if $\{b_{i_1}, \ldots, b_{i_s}\}$ is any subset of G with more than $[\frac{1}{2}n]$ elements, then $b_{i_1} \wedge \cdots \wedge b_{i_s} = 0$. Indeed, suppose that $b_{i_1} \wedge \cdots \wedge b_{i_s} \neq 0$ and let a be an atom such that $a \leqslant b_{i_1} \wedge \cdots \wedge b_{i_s}$. By Lemma 3 we infer that there is an index $h \leqslant s$ such that $a \wedge b_{i_h} = 0$, and then $a \wedge b_{i_1} \wedge \cdots \wedge b_{i_s} = 0$, a contradiction. Therefore, $b_1 \wedge \cdots \wedge b_t \wedge \bigwedge_{j \notin T} b_{i_j} = 0$ which implies that $d \wedge \bigwedge_{j \notin T} b_{i_j} = 0$. Then $a_i = b'_{i_1} \wedge \cdots \wedge b'_{i_{[n/2]}}$ and the proof is complete.

The same argument shows that if

$$q = \binom{n-t}{n-[\frac{1}{2}n]-t},$$

then D_{pq} can be generated with n elements, since

$$\binom{n}{n-[\frac{1}{2}n]} = \binom{n}{[\frac{1}{2}n]}.$$

(c) Suppose that D_{p1} can be generated with n elements, say $b_1, \ldots, b_n$. Arguing as in (b), we infer that there is an antichain of maximal size $A_1, \ldots, A_{p+1}$ of subsets of $\{1, \ldots, n\}$ such that $A_d \subseteq A_{p+1}$ and $A_d \nsubseteq A_i$, for $i = 1, \ldots, p$. Since $d > a_{p+1}$, we have that there is $j \in A_{p+1} \backslash A_d$. Suppose first that this antichain coincides with the set of all subsets of $\{1, \ldots, n\}$ of cardinal $[\frac{1}{2}n]$. It follows that $A_{p+1} \backslash \{j\}$ is a set with $[\frac{1}{2}n] - 1$ elements which is contained in exactly one set of cardinal $[\frac{1}{2}n]$. Therefore, $n - ([\frac{1}{2}n] - 1) = 1$, which implies that $n = [\frac{1}{2}n]$ and then $n = 0$, in contradiction with the fact that $p \neq 0$. A similar argument shows that if A_i has $n - [\frac{1}{2}n]$ elements for each $i \in \{1, \ldots, p+1\}$, then $p = 0$ and we obtain again a contradiction. $\square$

It follows from Theorems 2, 5 and 6 that we can determine the minimum number of variables needed to characterize the equational classes $\mathscr{D}_{p1}$, $\mathscr{D}_{p0}$ and the equational classes $\mathscr{D}_{p-1q+1}$ provided that $p \geqslant 1$ and $p+q$ is a natural number of the form $\binom{n}{[\frac{1}{2}n]}$. Although it seems to be difficult to find the minimum number of variables in the general case, it follows from Theorem 6(a) that this number can be only n or $n+1$.

Recall that a *monadic Boolean algebra* is a Boolean algebra endowed with a quantifier. Let us denote by $\mathscr{B}$ the variety of monadic Boolean algebras. It was shown in [8] that the lattice of equational classes of monadic Boolean algebras form also a chain of type $\omega + 1$ under inclusion $H_0 \subset H_1 \cdots \subset H_p \cdots \subset \mathscr{B}$, where, for each $p \in \omega$, H_p denotes the subvariety of $\mathscr{B}$ generated by the simple finite monadic algebra with p atoms.

It was also found in [8] an equation which determines H_p, for each $p \in \omega$. Note that Theorem 1(i) also provide us, for each proper subvariety of $\mathscr{B}$, an equation which determines it in which the symbol corresponding to the Boolean negation does not appear.

In [6] Lukas showed the following result:

For each $p \in \omega$, $p \neq 0$, the minimum number of variables needed in an identity characterizing H_p is the smallest n such that $2^n \geqslant p+1$.

On the other hand, we infer from the identity $2^n = \sum_{k=0}^{n} \binom{n}{k}$, the inequality $2^n \geqslant \binom{n}{[\frac{1}{2}n]} + 1$, for each $n \geqslant 1$. Therefore, it follows from Theorems 2 and 5 that for each $p \neq 0$, the minimum number of variables needed in an identity characterizing $\mathscr{D}_{p0}$ is greater or equal than the minimum number of variables needed in an identity characterizing H_p.

Acknowledgements

I wish to express my gratitude to Professor Roberto Cignoli for his advice during the preparation of this paper, in particular on the proof of Theorem 2, and to the referee, for his helpful comments.

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